MATH 2028 Honours Advanced Calculus II
2021-22 Term 1
Problem Set 5
due on Oct 25, 2021 (Monday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Gradescope on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Problems to hand in

1. Let $\Omega \subset \mathbb{R}^3$ be the open subset
$$\Omega = \{(x, y, z) \mid x^2 + y^2 + z^2 < a^2,\, z > 0\}.$$
Evaluate the integral $\int_\Omega z\, dV$ using spherical coordinates. Justify your answer carefully.

2. Let $\Omega \subset \mathbb{R}^2$ be the open subset lying in the first quadrant and bounded by the hyperbolas $xy = 1$, $xy = 2$ and the lines $y = x$, $y = 4x$. Evaluate the integral $\int_\Omega x^2y^3\, dA$.

3. Let $\Omega \subset \mathbb{R}^3$ be the open tetrahedron with vertices $(0, 0, 0)$, $(1, 2, 3)$, $(0, 1, 2)$ and $(-1, 1, 1)$. Evaluate the integral $\int_\Omega (x + 2y - z)\, dV$.

4. Let $\Omega \subset \mathbb{R}^2$ be the open subset bounded by $x = 0$, $y = 0$ and $x + y = 1$. Evaluate the integral $\int_\Omega \cos\left(\frac{x-y}{x+y}\right) dA$. *(Hint: note that the integrand is un-defined at the origin.)*

Suggested Exercises

1. Let $\Omega \subset \mathbb{R}^2$ be the open subset bounded by the curve $x^2 - xy + 2y^2 = 1$. Express the integral $\int_\Omega xy\, dA$ as an integral over the unit disk in $\mathbb{R}^2$ centered at the origin.

2. Find the volume of the solid region $\Omega \subset \mathbb{R}^3$ bounded below by the surface $z = x^2 + 2y^2$ and above by the plane $z = 2x + 6y + 1$ by expressing it as an integral over the unit disk in $\mathbb{R}^2$ centered at the origin.

3. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ and the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$.

4. Let $\Omega \subset \mathbb{R}^2$ be the open triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Evaluate the integral $\int_\Omega e^{(x-y)/(x+y)}\, dA$
   (a) using polar coordinates;
   (b) using the change of variables $u = x - y$, $v = x + y$.

5. Let $\Omega \subset \mathbb{R}^2$ be the open subset in the first quadrant bounded by $y = 0$, $y = x$, $xy = 1$ and $x^2 - y^2 = 1$. Evaluate the integral $\int_\Omega (x^2 + y^2)\, dA$ using the change of variables $u = xy$, $v = x^2 - y^2$.

6. Let $B^n(r)$ denote the closed ball of radius a in $\mathbb{R}^n$ centered at the origin.

(a) Show that $\mathrm{Vol}(B^n(r)) = \lambda_n r^n$ for some positive constant λ_n.

(b) Compute λ_1 and λ_2.

(c) Compute λ_n in terms of λ_{n-2}.

(d) Deduce a formula for λ_n for general n. *(Hint: consider two cases, according to whether n is even or odd.)*

Challenging Exercises

1. (a) Let $g : A \to \mathbb{R}^n$ be a C^1 map from an open subset $A \subset \mathbb{R}^n$. Denote the set

$$S = \{x \in A \mid \det Dg(x) = 0\}.$$

Prove that $g(S)$ has measure zero in $\mathbb{R}^n$.

(b) Use (a) to prove that the change of variables theorem still holds even if g is only a C^1 bijective map.